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Nonlinear electrorheological instability of two Rivlin–Ericksen elastico-viscous fluids

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Abstract

The behaviour of surface waves propagating between two Rivlin–Ericksen elastico-viscous fluids is examined. The investigation is made in the presence of a vertical electric field and a relative horizontal constant velocity. The influence of both surface tension and gravity force is taken into account. Due to the inclusion of streaming flow a mathematical simplification is considered. The viscoelastic contribution is demonstrated in the boundary conditions. From this point of view the approximation equations of motion are solved in the absence of viscoelastic effects. The solutions of the linearized equations of motion under nonlinear boundary conditions lead to derivation of a nonlinear equation governing the interfacial displacement and having damping terms with complex coefficients. This equation is accomplished by utilizing the cubic nonlinearity. The use of the Gardner–Morikawa transformation yields a simplified linear dispersion relation so that the periodic solution for the linear form is utilized. The perturbation analysis, in the light of the multiple scales in both space and time, leads to imposing the well-known nonlinear Schrödinger equation having complex coefficients. The stability criteria are discussed theoretically and illustrated graphically in which stability diagrams are obtained. Regions of stability and instability are identified for the electric fields versus the wavenumber for the wavetrain of the disturbance. Numerical calculations showed that the ratio of the dielectric constant plays a dual role in the stability criteria. The damping role for the viscosity coefficient is observed. The viscoelasticity coefficient plays two different roles. A stabilizing influence is observed through the linear scope and a destabilizing role in the nonlinear stability picture is seen.

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1. Introduction

Electrorheological fluids are special viscous liquids that are characterized by their ability to undergo significant changes in their mechanical properties when an electric field is applied. This property can be exploited in technological applications, e.g. actuators, clutches, shock absorbers and rehabilitation equipment to name a few. Winslow [1] is credited with the first observation of the behaviour of electrorheological fluids in 1949. Since then great strides have been made to overcome the impediments of early electroroheological fluids, such as the abrasive nature and the instability of the suspension and the enormous voltage requirements that are necessary for a significant change in the material properties. Nowadays existing electrorheological fluids, which make the above-mentioned devices possible, are the result of intensive efforts to manufacture materials without these impediments.

In recent years, a great deal of interest has been focused on the understanding of the rheological effects occurring in the flow of non-Newtonian fluids. Now this problem appears to be of special interest in oil reservoir engineering, where increasing interest is shown in the possibility of improving oil recovery efficiency from water flooding projects through mobility control with non-Newtonian displacing fluids. Consequently, it has become essential to have an adequate understanding of the rheological effect of non-Newtonian displacing and displaced fluids in an oil displacement mechanism. Many technological processes involve the parallel flow of fluids of different viscosity, elasticity, density and with different horizontal velocities. Such flows exist in pacbed bed reactors in the chemical industry, petroleum engineering, plasma physics and in many other processes.

The interface between two bulk media can be observed in a large number of technological areas: mechanical and chemical engineering, the petroleum industry, combustion and so on. Very often the interfaces are considered as surfaces which separate the adjacent phases or as places allowing the exchange of mass, momentum and energy. It can also be imagined that interfaces are composed of a material medium and therefore have their own material properties.

Interfaces separating two material phases occur in numerous areas of application. In fluid dynamics there are interfaces separating two fluid phases, as in thin liquid films, droplets, spreading, nucleate boiling and the fronts separating nematic and isotropic liquid crystals. Interfaces between solids and fluids are important in crystal growth and deformable porous media. In addition, interfaces may separate multiphase media as when a bubbly or particle-laden fluid abuts a second fluid, in combustion involving mixtures or in porous media. The last occurs in combustion synthesis of materials, in waste incinerators, in underground oil recovery and in living tissues. Interfacial dynamics is the controlling factor in a whole spectrum of high-technology industrial-product manufacture and simultaneously the inspiration for fundamental scientific investigation. In all the cases mentioned above the phase interface is a moving free boundary whose shape, position and dynamics are to be determined. Each of these represents nonlinear problems involving instability and bifurcation phenomena, wavelength selection and chaotic behaviour. The above examples exhibit the whole gamut of dynamical phenomena including coalescence and rupture, melting, freezing, sedimentation, agglomeration, chemical reactions, moving contact lines, cellular, pulsatile, dendritic and spiral patterns, chaotic behaviour and all the particular idiosyncrasies of suspensions, colloids, foams and granular or porous materials.

The instability of the plane between two superposed fluids with a relative horizontal velocity is called the Kelvin–Helmholtz instability. The Kelvin–Helmholtz instability due to shear flow in stratified fluids has attracted the attention of many researchers because of its determinant influence on the stability of planetary and stellar atmospheres and in practical applications. The study of the Kelvin–Helmholtz instability has a long history in hydrodynamics. The Kelvin–Helmholtz problem is the instability of two inviscid layers of fluid separated by an immiscible interface and has been studied at length, for example in the books by Craik [2], Drazin and Reid [3] and Chandrasekhar [4]. Kelvin–Helmholtz instability arises due to velocity shear in the fluid. Chandrasekhar [4] has made a comprehensive survey of the hydromagnetic version of this instability. An excellent review of the Kelvin–Helmholtz instability has been presented by Gerwin [5]. The Kelvin–Helmholtz instability is of interest in investigating a variety of space, astrophysical and geophysical situations. In the Kelvin–Helmholtz model, the effect of streaming is destabilizing in the linear sense (Chandrasekhar [4]). Lyon [6] added the effect of compressibility and applied electric field, but neglected surface tension. Melcher [7] discussed the influence of both vertical and horizontal electric fields on Kelvin–Helmholtz for incompressible flow in the presence of surface tension effects. The nonlinear development of the Kelvin–Helmholtz instability in two semi-infinite pure fluids has been considered by Drazin [8], Nayfeh and Saric [9] for the case where the amplitude of an unstable wave is uniform in space and growing only in time. Weissman [10] has studied the nonlinear development of the Kelvin–Helmholtz instability for packets of waves in which the amplitude is a function of space as well as time.

On the other hand, electrohydrodynamics is the field of the mechanics of continua that studies the motion of media interacting with an electric field. Such an interaction takes place as a result of the action of the Coulomb force upon a medium, or as a result of the work of an electric field in the flow of currents. The motion of a medium gives rise to re-distribution of a volume charge, which results in changing the electric field and, hence, the force acting on a medium. In the majority of problems under consideration, electric fields are specified by external sources. Such a situation takes place during operation of electrohydrodynamic generators, pumps, separators, filters and other devices. This subject is treated in a vast selection of literature. The last studied is the class of electrohydrodynamic problems in which the electric field or electric charges arise as a result of a contact between media of different nature: liquid–solid body, liquid–gas or two different liquids (see [11–16]).

The nonlinear electrohydrodynamic Kelvin–Helmholtz instability for inviscid fluids stressed by a normal electric field was discussed earlier by Elshehawey [17] and Mohamed and Elshehawey [18]. They assumed that the applied electric field has components perpendicular and tangential to the fluid interface. Elhefnawy [19] discussed the same problem for the Marangoin effect in the presence of the components of perpendicular and tangential electric fields. Mohamed *et al* [20] studied the influence of mass and heat transfer on the Kelvin–Helmholtz instability. Malik and Singh [21] investigated the nonlinear Kelvin–Helmholtz properties of (2 + 1)-dimensional wave packets propagating at the interface of two superposed ferrofluids. They considered the fluids to be moving with uniform speeds parallel to the common interface and subject to a tangential magnetic field. They derived a nonlinear equation that governs the evolution of the amplitude of the system. In one of the most recent works on this subject the effects of periodic body forces were discussed. The effect of a time-dependent acceleration in the presence of a tangential magnetic field on the nonlinear stability of Kelvin–Helmholtz wave has been discussed by El-Dib [22].

The study of viscoelastic fluids has become increasingly important in the last few years. This is mainly due to their many applications in petroleum drilling, manufacture of foods and paper and many other similar activities. The surface between two fluids is of special importance owing to its application to many engineering problems. A generalization of the Kelvin–Helmholtz instability for viscoelastic flow is a very difficult problem. This difficulty arises as the two viscoelastic fluids are in relative tangential motion. The Kelvin–Helmholtz model for viscous, flow or for viscoelastic flow represents an ill-posed problem. This is because

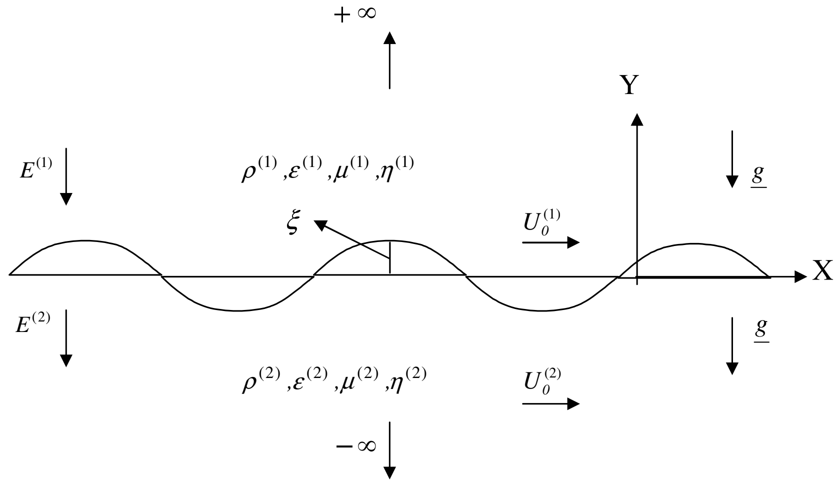

Figure 1. Sketch for the system under consideration. The y-axis is taken vertically upwards. The x-axis is taken horizontally at the flat interface. $E^{(j)}$ is the normal electric field in the fluid layers. $\rho^{(1)}$ and $\rho^{(2)}$ are the fluid densities, $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ are the dielectric constants, $\eta^{(1)}$ and $\eta^{(2)}$ are the viscoelasticity coefficients, $\mu^{(1)}$ and $\mu^{(2)}$ are the viscosity coefficients for the fluid media, $U_0^{(1)}$ and $U_0^{(2)}$ are constant horizontal velocities, ξ is the deflection of the interface and g is the gravitational acceleration.

the tangential velocity could have a non-vanishing jump. In order to study this problem for viscous flow, Feng [23] restricts himself to the case of weak viscous effects. This weakness is regarded such that the viscous effect appears at the interface and gradually decreases so that it is negligible in the bulk [23–25]. Its contribution will be demonstrated in the boundary conditions. El-Dib and Matoog [26] have studied the Kelvin–Helmholtz instability for the Maxwellian fluid sheet. They discussed the linear instability under the influence of a periodic electric field. They applied the Feng [23] treatment where the viscoelastic contribution has been demonstrated through the normal stress boundary condition. Recently, in 2002, El-Dib and Moatimid [27] investigated the stability of a viscoelastic cylindrical interface acted upon by an oscillating azimuthal magnetic field in the presence of free-surface currents. Weak viscoelastic effects were taken into consideration. They applied the same procedure as in [23–26], in which the viscoelastic contributions are in the boundary conditions.

The present work is concerned with simple viscoelastic fluid flows with constant relative velocities. A generalization of the nonlinear Kelvin–Helmholtz instability for Rivlin–Ericksen flow under the effect of an electric field is the goal of this study. In this work, the viscoelastic contribution will be demonstrated through the boundary conditions. However, one must address some additional complexities that are principally due to the interactions between viscoelastic fluids and an electric field through nonlinear analysis. In general, viscoelastic flow problems involving two fluids are exceedingly difficult to solve even when the shapes of the interface are assumed to be known. However, in this research we have concentrated on a weak nonlinear scheme that is based on the idea of solutions for linearizing the electrohydrodynamic equations of motion under nonlinearizing boundary conditions.

2. Basic equations

The flow system specifically associated with the stability problem with which we are concerned here is shown in figure 1. A parallel flow of two immiscible viscoelastic fluids in an infinite, fully saturated, uniform and homogeneous medium is shown in the figure. The two fluids are incompressible and have constant properties. The interface between the fluids is assumed to be well defined and initially flat and forms the plane $y = 0$.

The fluids have a viscoelastic nature described by the following constitutive relation:

$$\tau_{ij} = -\pi\delta_{ij} + \left[\mu + \eta\left(\frac{\partial}{\partial t} + V_k\frac{\partial}{\partial x_k}\right)\right]\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) \tag{1}$$

where τ_{ij} is the hydrodynamic stress tensor, δ_{ij} is the Kronecker delta, π is the hydrostatic pressure, μ is the coefficient of viscosity, η is the viscoelasticity coefficient and $\underline{V}$ is the fluid velocity vector. The two fluids are streaming with constant horizontal velocities $U_0^{(1)}$ and $U_0^{(2)}$. The superscripts (1) and (2) refer to the upper fluid and lower fluid, respectively, $\rho^{(1)}$ and $\rho^{(2)}$ are the fluid densities and $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ are the dielectric constants. The system is considered to be influenced by gravity in the negative y-direction. The fluids are subjected to external vertical electric fields $E^{(1)}$ and $E^{(2)}$ acting in the negative y-direction. There are no volume charges in the layers. In addition there are no free-surface charges at the interface between fluids. The electric Maxwell stress tensor [7] is given by

$$M_{ij} = \varepsilon E_i E_j - \tfrac{1}{2}\varepsilon E^2\delta_{ij}. \tag{2}$$

The total stress tensor can be described as

$$\sigma_{ij} = \tau_{ij} + M_{ij}. \tag{3}$$

As the initial state of the system, we assume that both fluid phases are immiscible and have a common flat interface at $y = 0$. The distribution equilibrium of the interface between both electrified fluid phases has been established. We are interested in the interfacial response of the two phases. Due to the very intricate nature of the problem for the generalization of the Kelvin–Helmholtz flow of non-Newtonian dielectric fluids, we confine the analysis to considering weak viscoelastic effects. The introduction of the weak viscous effects is customary in the case of a Newtonian fluid [23–25] and should be understood in a manner similar to a viscoelastic problem, which makes problem formulation much easier to handle, since relative motions are included. Moreover, the same technique has been successfully applied to a viscoelastic fluid of Maxwellian type under the linear perturbation theory by El-Dib and Matoog [26] and El-Dib and Moatimid [27].

The assumption that viscoelasticity produces only a thin, weak vertical layer at the surface of separation allows the motion to be supposed remaining irrotational throughout the bulk of the fluids. Thus the derivations in this problem deal completely with potential flow so that complicated manipulation of the boundary-layer equations for the weak vertical flow can be avoided. Since the field equation governing the irrotational flow is the Laplace equation, modifying the boundary conditions at the surface should be an acceptable means of including the small viscoelastic effects. The non-zero irrotational tangential stress near the surface drags a thin vertical layer along, making a modification to the velocity field. In the present theory, viscoelastic effects can be formulated from the normal stress boundary condition. Therefore, and in view of the viscoelastic approximation considered here, the governing equation for the bulk of the fluid phase is

$$\rho\left[\frac{\partial \underline{V}}{\partial t} + (\underline{V}\cdot\nabla)\underline{V}\right] = -\nabla\pi - \rho g\underline{e}_y \tag{4}$$

where ∇ is the gradient operator, $\underline{V}$ is the fluid velocity satisfied by the following continuity equation for incompressible fluids:

$$\nabla\cdot\underline{V} = 0. \tag{5}$$

The distribution equilibrium of the interface between both electrified fluid phases has been established. We are interested in the interfacial response of the two phase system after a disturbance of the equilibrium configuration. The surface deflection is expressed by

$$y = \xi(x, t). \tag{6}$$

If the surface is defined as the locus of points satisfying the relation $S(x, y, t) = 0$, then the unit normal vector to the interface is given by

$$\underline{n} = \frac{\nabla s}{|\nabla s|} = \left(-\frac{\partial \xi}{\partial x}\underline{e}_x + \underline{e}_y\right)\left[1 + \left(\frac{\partial \xi}{\partial x}\right)^2\right]^{-1/2} \tag{7}$$

where the vectors $\underline{e}_x$ and $\underline{e}_y$ are the unit vectors in the x- and y-directions. All disturbances are assumed to be two dimensional. By two dimensional we mean that the flow field depends on the horizontal direction of propagation, i.e. the x-axis, and the vertical y-direction.

As a result of perturbation, the initial fluid velocity increases and hence equation (5) allows the introduction of the stream function $\psi(x, y, t)$ such that the total velocity is defined as

$$\underline{V} = \left(U_0 - \frac{\partial \psi}{\partial y}\right)\underline{e}_x + \frac{\partial \psi}{\partial x}\underline{e}_y. \tag{8}$$

This should be substituted into the above system and an equivalent boundary value problem to be solved for ψ. This will be obtained later. It is convenient to ensure that $\psi(x, y, t)$ is a finite function presented due to the interface disturbance and far from the interface. Its influence is neglected. Therefore, both the partial derivatives for $\psi(x, y, t)$, with respect to x and y, must vanish as $y \to \pm\infty$. In other words, far from the interface, the stream functions are assumed to be

$$\psi^{(1)}(x, +\infty, t) = \psi_\infty^{(1)} \qquad \text{and} \qquad \psi^{(2)}(x, -\infty, t) = \psi_\infty^{(2)} \tag{9}$$

where $\psi_\infty^{(1)}$ and $\psi_\infty^{(2)}$ are two different finite constants. At the fluid interface, it is required that both the horizontal and the vertical components for the fluid velocity field satisfy an equation expressing the assumed material character of the dividing surface. Such an equation is

$$\frac{\partial \psi^{(j)}}{\partial x} + \frac{\partial \xi}{\partial x}\frac{\partial \psi^{(j)}}{\partial y} = \frac{\partial \xi}{\partial x}U_0^{(j)} + \frac{\partial \xi}{\partial t} \qquad j = 1, 2 \quad y = \xi. \tag{10}$$

The electrically insulating fluid justifies the stationary form of the Maxwell equations, which are reduced to the Laplace equation for the electric potentials $\phi^{(j)}$ in each of the two different regions. The scalar electric potentials are defined by

$$\underline{E}^{(j)} = -E^{(j)}\underline{e}_y - \nabla\phi^{(j)} \qquad j = 1, 2 \tag{11}$$

where the suffix j denotes the upper or the lower medium. The potential ϕ has to satisfy Laplace's equation,

$$\nabla^2\phi^{(j)} = 0 \tag{12}$$

and subject to the following boundary conditions:

(1) $\nabla\phi$, vanishes far from the fluid interface

$$\nabla\phi^{(j)}(x, \pm\infty, t) = 0. \tag{13}$$

(2) At the dividing surface the potential $\phi(x, y, t)$ must satisfy the following conditions:

(i) The jump in the tangential component of the electric field is zero across the interface $y = 0$. Thus,

$$\frac{\partial \xi}{\partial x}\left[(E^{(1)} - E^{(2)}) + \frac{\partial}{\partial y}(\phi^{(1)} - \phi^{(2)})\right] + \frac{\partial}{\partial x}(\phi^{(1)} - \phi^{(2)}) = 0 \qquad y = \xi. \tag{14}$$

(ii) The continuity of the normal component of the electric displacement at the surface of separation yields

$$\frac{\partial \xi}{\partial x}\frac{\partial}{\partial x}(\varepsilon^{(1)}\phi^{(1)} - \varepsilon^{(2)}\phi^{(2)}) - \frac{\partial}{\partial y}(\varepsilon^{(1)}\phi^{(1)} - \varepsilon^{(2)}\phi^{(2)}) = 0 \qquad y = \xi. \tag{15}$$

Since we are dealing with the case of purely polarization effects, i.e., there are no free charges on the interface between the two fluids, then [7]

$$\varepsilon^{(1)}E^{(1)} - \varepsilon^{(2)}E^{(2)} = 0. \tag{16}$$

The pure equilibrium configuration gives

$$\pi_0^{(j)}(x, y, t) = -\rho^{(j)}gy - U_0^{(j)}x + C_0^{(j)} \qquad j = 1, 2 \tag{17}$$

where π_0 is the equilibrium pressure and C_0 is the constant of integration. The balance of the normal stress tensor at the interface leads to

$$C_0^{(1)} - C_0^{(2)} = \tfrac{1}{2}\big(\varepsilon^{(1)}E^{(1)^2} - \varepsilon^{(2)}E^{(2)^2}\big). \tag{18}$$

The solution of the equations of motion cited earlier is accomplished by utilizing the convenient boundary conditions. At the boundary between fluids, the fluids and the electrical stresses must be balanced. The components of these consist of the hydrodynamic pressure, viscous stresses, surface tension stresses and electrical stresses. The electrical stresses result from the dielectric forces [7, 29].

The normal component of the stress tensor σ_{ij} is discontinuous at the interface because of surface tension [30], i.e.

$$\left(\frac{\partial \xi}{\partial x}\right)^2\big(\sigma_{11}^{(1)} - \sigma_{11}^{(2)}\big) - 2\frac{\partial \xi}{\partial x}\big(\sigma_{21}^{(1)} - \sigma_{21}^{(2)}\big) + \big(\sigma_{22}^{(1)} - \sigma_{22}^{(2)}\big) = -\sigma_T\frac{\partial^2 \xi}{\partial x^2}\left[1 + \left(\frac{\partial \xi}{\partial x}\right)^2\right]^{-1/2}$$

$$y = \xi \tag{19}$$

where σ_T is the surface tension coefficient. Expanding the above total stress tensor (3) for two-dimensional Cartesian coordinates, and substituting into condition (19) gives

$$\begin{aligned}
&\pi^{(2)} - \pi^{(1)} + (\rho^{(1)} - \rho^{(2)})g\xi + \sigma_T\frac{\partial^2 \xi}{\partial x^2}\left[1 + \left(\frac{\partial \xi}{\partial x}\right)^2\right]^{-3/2} + \left[1 - \left(\frac{\partial \xi}{\partial x}\right)^2\right]\left[1 + \left(\frac{\partial \xi}{\partial x}\right)^2\right]^{-1} \\
&\quad \times\left\{2\left(\mu^{(1)} + \eta^{(1)}\left(\frac{\partial}{\partial t} + U_0^{(1)}\frac{\partial}{\partial x}\right)\right)\frac{\partial^2\psi^{(1)}}{\partial x\partial y} - 2\left(\mu^{(2)} + \eta^{(2)}\left(\frac{\partial}{\partial t} + U_0^{(2)}\frac{\partial}{\partial x}\right)\right)\right. \\
&\quad \times\frac{\partial^2\psi^{(2)}}{\partial x\partial y} - \frac{1}{2}\varepsilon^{(1)}\left[\left(\frac{\partial\phi^{(1)}}{\partial x}\right)^2 - \left(\frac{\partial\phi^{(1)}}{\partial y}\right)^2\right] + \frac{1}{2}\varepsilon^{(2)}\left[\left(\frac{\partial\phi^{(2)}}{\partial x}\right)^2 - \left(\frac{\partial\phi^{(2)}}{\partial y}\right)^2\right] \\
&\quad \left.-\left(\varepsilon^{(1)}E^{(1)}\frac{\partial\phi^{(1)}}{\partial y} - \varepsilon^{(2)}E^{(2)}\frac{\partial\phi^{(2)}}{\partial y}\right)\right\} - 2\frac{\partial \xi}{\partial x}\left[1 + \left(\frac{\partial \xi}{\partial x}\right)^2\right]^{-1} \\
&\quad \times\left\{\left(\mu^{(1)} + \eta^{(1)}\left(\frac{\partial}{\partial t} + U_0^{(1)}\frac{\partial}{\partial x}\right)\right)\left(\frac{\partial^2\psi^{(1)}}{\partial x^2} - \frac{\partial^2\psi^{(1)}}{\partial y^2}\right)\right. \\
&\quad -\left(\mu^{(2)} + \eta^{(2)}\left(\frac{\partial}{\partial t} + U_0^{(2)}\frac{\partial}{\partial x}\right)\right)\left(\frac{\partial^2\psi^{(2)}}{\partial x^2} - \frac{\partial^2\psi^{(2)}}{\partial y^2}\right)
\end{aligned}$$

$$-\left(\varepsilon^{(1)}E^{(1)}\frac{\partial\phi^{(1)}}{\partial x}-\varepsilon^{(2)}E^{(2)}\frac{\partial\phi^{(2)}}{\partial x}\right)+\varepsilon^{(1)}\left(\frac{\partial\phi^{(1)}}{\partial x}\right)\left(\frac{\partial\phi^{(1)}}{\partial y}\right)$$
$$-\varepsilon^{(2)}\left(\frac{\partial\phi^{(2)}}{\partial x}\right)\left(\frac{\partial\phi^{(2)}}{\partial y}\right)\Bigg\}=0 \qquad y=\xi. \tag{20}$$

The boundary conditions represented here are prescribed at the interface $y=\xi(x,t)$. As the interface is deformed all variables are slightly perturbed from their equilibrium values. Because the interfacial displacement is small, the boundary conditions on perturbation interfacial variables need to be evaluated at the equilibrium position rather than at the interface. Therefore, it is necessary to express all the physical quantities involved in terms of Maclaurin series about $y=0$.

3. Nonlinear equation governing the interfacial displacement

To solve the linearization equation for the fluid phases under consideration two-dimensional finite disturbances are introduced into the equation of motion and continuity equation as well as nonlinear boundary conditions. In accordance with the standard Fourier decomposition, we may similarly assume that the bulk solutions are of the form

$$\psi^{(j)}(x,y,t)=\hat{\psi}^{(j)}(y,t)\,\mathrm{e}^{\mathrm{i}kx} \tag{21}$$

$$\pi^{(j)}(x,y,t)=\hat{\pi}^{(j)}(y,t)\,\mathrm{e}^{\mathrm{i}kx} \tag{22}$$

$$\phi^{(j)}(x,y,t)=\hat{\phi}^{(j)}(y,t)\,\mathrm{e}^{\mathrm{i}kx} \qquad j=1,2 \tag{23}$$

where k is the wavenumber, which is assumed to be real and positive, $\mathrm{i}=\sqrt{-1}$. Substituting from (23) into the Laplace equation (12), for using the boundary conditions (14) and (15), the resulting solutions are related to the interfacial displacement ξ as

$$\phi^{(1)}(x,y,t)=-\mathrm{i}E^{(1)}\frac{(\varepsilon^{(1)}-\varepsilon^{(2)})}{(\varepsilon^{(1)}+\varepsilon^{(2)})}\frac{\xi_x}{k(1+\mathrm{i}\xi_x)}\,\mathrm{e}^{-ky} \qquad y>0 \tag{24}$$

$$\phi^{(2)}(x,y,t)=\mathrm{i}E^{(2)}\frac{(\varepsilon^{(1)}-\varepsilon^{(2)})}{(\varepsilon^{(1)}+\varepsilon^{(2)})}\frac{\xi_x}{k(1-\mathrm{i}\xi_x)}\,\mathrm{e}^{ky} \qquad y<0 \tag{25}$$

where the subscript for the variable ξ refers to partial derivatives. The above distribution for the electric potential functions $\phi^{(1)}(x,y,t)$ and $\phi^{(2)}(x,y,t)$ contains nonlinear terms in the elevation parameter ξ. This nonlinearity is due to the use of the boundary conditions (14) and (15) without dropping the nonlinear terms. As the nonlinear terms are ignored, the linear profile arises and is equivalent to those obtained earlier by Melcher [6].

Continuity equation (5) will be used to eliminate the pressure from the linearized form of equation of motion (4). Then with the help of (8) the equation of motion is reduced to the Laplace equation in the stream function $\psi^{(j)}(x,y,t)$. Substituting (21) into the resulting Laplace equation and in the light of (9) and (10), we obtain

$$\psi^{(1)}(x,y,t)=\frac{\left(U_0^{(1)}\xi_x+\xi_t\right)}{k(\mathrm{i}-\xi_x)}\,\mathrm{e}^{-ky} \qquad y>0 \tag{26}$$

$$\psi^{(2)}(x,y,t)=\frac{\left(U_0^{(2)}\xi_x+\xi_t\right)}{k(\mathrm{i}+\xi_x)}\,\mathrm{e}^{ky} \qquad y<0. \tag{27}$$

Accordingly, the distribution of the pressure function π in the two fluid phases should be evaluated in the form

$$\pi^{(1)}(x,y,t)=\frac{-\mathrm{i}\rho^{(1)}}{k(\mathrm{i}-\xi_x)}\left(U_0^{(1)}\xi_{xt}+\xi_{tt}\right)\mathrm{e}^{-ky} \qquad y>0 \tag{28}$$

$$\pi^{(2)}(x, y, t) = \frac{\mathrm{i}\rho^{(2)}}{k(\mathrm{i}+\xi_x)}\big(U_0^{(2)}\xi_{xt} + \xi_{tt}\big)\,\mathrm{e}^{ky} \qquad y < 0. \tag{29}$$

Both the stream function distribution and the pressure distribution contain nonlinear terms in the elevation parameter ξ. This nonlinearity is due to the use of boundary conditions involving nonlinear terms. As the nonlinear terms are ignored, the linear profile arises, equivalent to those obtained earlier by El-Dib and Matoog [26] in which they discussed the linear stability for the Kelvin–Helmholtz of viscoelastic flow.

At the boundary between fluids, the fluids and the electrical stress must be balanced. The components of these stresses consist of hydrodynamic pressure, viscoelastic stresses, velocities, surface tension stresses and electrical stresses. Inserting (26) through (29) into the normal stress tensor (20), to replace the dependence on the electric potential $\phi^{(j)}$, the stream function $\psi^{(j)}$ and the pressure function $\pi^{(j)}(x, y, t)$ by their equivalent in terms of elevation parameter ξ, results in a very complicated nonlinear equation in the elevation ξ. Keeping in mind that the elevation function ξ is small, the use of the binomial expansion is convenient. For terms to the third order of ξ, the following nonlinear equation in the interfacial displacement arises:

$$\begin{aligned}(\rho^{(1)} + \rho^{(2)})\xi_{tt} &+ 2\mathrm{i}k(\eta^{(1)} + \eta^{(2)})\xi_{xtt} + \mathrm{i}k\sigma_E\xi_x - kg(\rho^{(1)} - \rho^{(2)})\xi \\ &+ 4\mathrm{i}k\big(\eta^{(1)}U_0^{(1)} + \eta^{(2)}U_0^{(2)}\big)\xi_{xxt} + \big[2\mathrm{i}k(\mu^{(1)} + \mu^{(2)}) + \big(\rho^{(1)}U_0^{(1)} + \rho^{(2)}U_0^{(2)}\big)\big]\xi_{xt} \\ &+ k\big[\sigma_T + 2\mathrm{i}\big(\mu^{(1)}U_0^{(1)} + \mu^{(2)}U_0^{(2)}\big)\big]\xi_{xx} + N_2 + N_3 + \cdots = 0\end{aligned} \tag{30}$$

where σ_E refers to the electric term, which can be defined as

$$\sigma_E = E^{(1)}E^{(2)}\frac{(\varepsilon^{(1)} - \varepsilon^{(2)})^2}{(\varepsilon^{(1)} + \varepsilon^{(2)})}. \tag{31}$$

N_2 and N_3 are, respectively, denoted by the nonlinear second-order terms and third-order terms in the elevation parameter ξ as well as in its partial derivatives with respect to the variables t and x. These nonlinear terms are due to non-negligible the nonlinear terms from the boundary conditions cited above. The structure of these nonlinear terms is as given below

$$\begin{aligned}N_2 = &-\mathrm{i}(2k^2(\eta^{(1)} - \eta^{(2)}) + (\rho^{(1)} - \rho^{(2)}))\,\xi_x\xi_{tt} + 2\mathrm{i}(\eta^{(1)} - \eta^{(2)})\xi_x\xi_{xxtt} - k\sigma_E\frac{\varepsilon^{(1)} - \varepsilon^{(2)}}{\varepsilon^{(1)} + \varepsilon^{(2)}}\xi_x^2 \\ &- \mathrm{i}\big[2\mathrm{i}k(\mu^{(1)} - \mu^{(2)}) + 4k^2\big(\eta^{(1)}U_0^{(1)} - \eta^{(2)}U_0^{(2)}\big) + \big(\rho^{(1)}U_0^{(1)} - \rho^{(2)}U_0^{(2)}\big)\big]\xi_x\xi_{xt} \\ &- 2\mathrm{i}k\left[\mathrm{i}\big(\mu^{(1)}U_0^{(1)} - \mu^{(2)}U_0^{(2)}\big) + k\left(\eta^{(1)}U_0^{(1)^2} - \eta^{(2)}U_0^{(2)^2}\right)\right]\xi_x\xi_{xx} \\ &- 2\mathrm{i}k^2\big(\mu^{(1)}U_0^{(1)} - \mu^{(2)}U_0^{(2)}\big)\xi_x^2 + \big[2k\big(\eta^{(1)}U_0^{(1)} - \eta^{(2)}U_0^{(2)}\big) \\ &+ 2\mathrm{i}\big(\mu^{(1)}U_0^{(1)} - \mu^{(2)}U_0^{(2)}\big)\big]\,\xi_x\xi_{xxx} + 2\mathrm{i}\left(\eta^{(1)}U_0^{(1)^2} - \eta^{(2)}U_0^{(2)^2}\right)\xi_x\xi_{xxxx} \\ &+ 2\big[\mathrm{i}(\mu^{(1)} - \mu^{(2)}) + 2k\big(\eta^{(1)}U_0^{(1)} - \eta^{(2)}U_0^{(2)}\big)\big]\,\xi_x\xi_{xxt} \\ &+ 2k(\eta^{(1)} - \eta^{(2)})\,\xi_x\xi_{xtt} + 4\mathrm{i}\big(\eta^{(1)}U_0^{(1)} - \eta^{(2)}U_0^{(2)}\big)\,\xi_x\xi_{xxxxt}\end{aligned} \tag{32}$$

$$\begin{aligned}N_3 = &\tfrac{1}{2}k\left[\sigma_T - 4\mathrm{i}\big(\mu^{(1)}U_0^{(1)} + \mu^{(2)}U_0^{(2)}\big) - 4k\left(\eta^{(1)}U_0^{(1)^2} + \eta^{(2)}U_0^{(2)^2}\right)\right]\xi_x^2\xi_{xx} \\ &- 2\mathrm{i}k(\eta^{(1)} + \eta^{(2)})\,\xi_x^2\xi_{xtt} + \big[-2\mathrm{i}k(\mu^{(1)} + \mu^{(2)}) + \big(\rho^{(1)}U_0^{(1)} + \rho^{(2)}U_0^{(2)}\big) \\ &- 4k^2\big(\eta^{(1)}U_0^{(1)} + \eta^{(2)}U_0^{(2)}\big)\big]\,\xi_x^2\xi_{xt} + 2\left[\big(\mu^{(1)}U_0^{(1)} + \mu^{(2)}U_0^{(2)}\big)\right. \\ &\left.- \mathrm{i}k\left(\eta^{(1)}U_0^{(1)^2} + \eta^{(2)}U_0^{(2)^2}\right)\right]\xi_x^2\xi_{xxx} + 2\left(\eta^{(1)}U_0^{(1)^2} + \eta^{(2)}U_0^{(2)^2}\right)\xi_x^2\xi_{xxxx}\end{aligned}$$

$$+2\left[(\mu^{(1)}+\mu^{(2)})-2\mathrm{i}k\left(\eta^{(1)}U_0^{(1)}+\eta^{(2)}U_0^{(2)}\right)\right]\xi_x^2\xi_{xxt}$$
$$+4\left(\eta^{(1)}U_0^{(1)}+\eta^{(2)}U_0^{(2)}\right)\xi_x^2\xi_{xxxt}+2(\eta^{(1)}+\eta^{(2)})\,\xi_x^2\xi_{xxtt}+2kg(\rho^{(1)}-\rho^{(2)})\,\xi_x^2\xi. \tag{33}$$

The absence of these nonlinear terms leads to the production of the linear form for the problem. It is seen that the direct treatment of the linear problem, for elevation having periodic spatial and periodic temporal nature, yields a linear dispersion relation with complex coefficients. To relax this complexity for the linear form (which represents the basic state for nonlinear perturbations), it is convenient to use the following transformation:

$$X = x-\mu^* t \qquad T = t \qquad \mu^* = \frac{\mu^{(1)}U_0^{(1)}+\mu^{(2)}U_0^{(2)}}{\mu^{(1)}+\mu^{(2)}}. \tag{34}$$

Consequently, the partial derivatives with respect to t and x should be changed to

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial T}-\mu^*\frac{\partial}{\partial X} \qquad \text{and} \qquad \frac{\partial}{\partial x}=\frac{\partial}{\partial X}. \tag{35}$$

At this stage, the above nonlinear system (30) reduces to

$$\Bigg\{\left[\left(\rho^{(1)}+\rho^{(2)}\right)+2\mathrm{i}k(\eta^{(1)}+\eta^{(2)})\frac{\partial}{\partial X}\right]\left(\frac{\partial}{\partial T}-\mu^*\frac{\partial}{\partial X}\right)^2+\left[\left(\rho^{(1)}U_0^{(1)}+\rho^{(2)}U_0^{(2)}\right)\right.$$
$$\left.+2\mathrm{i}k\left(\eta^{(1)}U_0^{(1)}+\eta^{(2)}U_0^{(2)}\right)\frac{\partial}{\partial X}\right]\frac{\partial}{\partial X}\left(\frac{\partial}{\partial T}-\mu^*\frac{\partial}{\partial X}\right)+k\sigma_T\frac{\partial^2}{\partial X^2}$$
$$-\mathrm{i}k\sigma_E\frac{\partial}{\partial X}+kg(\rho^{(1)}-\rho^{(2)})\Bigg\}\,\xi(X,T)+\tilde{N}_2+\tilde{N}_3=0 \tag{36}$$

where $\tilde{N}_j$ is produced from N_j, under the transformations (34) and (35). The existence of a harmonic wavetrain in a dispersive medium requires

$$\xi=\gamma\,\mathrm{e}^{\mathrm{i}kX-\mathrm{i}(\omega+\nu-k\mu^*)T}+\mathrm{c.c.}, \tag{37}$$

where

$$\nu=\frac{k\left(\left(\rho^{(1)}U_0^{(1)}+\rho^{(2)}U_0^{(2)}\right)-2k^2\left(\eta^{(1)}U_0^{(1)}+\eta^{(2)}U_0^{(2)}\right)\right)}{2\left((\rho^{(1)}+\rho^{(2)})-2k^2(\eta^{(1)}+\eta^{(2)})\right)}. \tag{38}$$

The correspondence between the wavenumber k and frequency ω leads to the following linear dispersion function:

$$D(\omega,k)=((\rho^{(1)}+\rho^{(2)})-2k^2(\eta^{(1)}+\eta^{(2)}))(\nu^2-\omega^2)-k(k^2\sigma_T-k\sigma_E-g(\rho^{(1)}-\rho^{(2)})). \tag{39}$$

Equation (36) is more general than the characteristic equation obtained earlier by Chandrasekhar [4]. In addition to the nonlinear contributions, this dispersion relation includes the viscoelastic effects as well as the electric field influence.

4. Linear problem as a limiting case

The linearized form of equation (36) should arise when the higher orders of ξ are ignored. To this end, equation (36) is reduced to

$$L\left(\frac{\partial}{\partial T},\frac{\partial}{\partial X}\right)\xi=0 \tag{40}$$

where L is a linear operator involving the partial derivative of $\partial/\partial \mathrm{T}$ and $\partial/\partial \mathrm{X}$. We first study equation (40) and then return to equation (36) to incorporate higher order dispersive effects. Suppose we require a uniform monochromatic wavetrain solution to (40) in the form of (37). Consequently, the frequency ω is determined in the form

$$\omega^2 = \nu^2 + \frac{k(k^2\sigma_T - k\sigma_E - g(\rho^{(1)} - \rho^{(2)}))}{2k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)})}. \tag{41}$$

It should be noted that (41) represents the linear dispersion relation for surface waves propagating between two superposed streaming fluids having small viscoelastic influence. It is noted that the viscosity coefficients are not included in this dispersion relation. The implication for the damping terms should be in nonlinear terms. As the parameter $\eta^{(j)}$ tends to zero, the results will refer to the pure Newtonian Kelvin–Helmholtz instability for the weak viscous effect. When $U_0^{(j)} \to 0$, the Rayleigh–Taylor model arises. Thus, both the parameters ν and μ^* should be zero. To this end, the linear dispersion relation that governs the linear case for the Rayleiegh–Taylor instability can be derived from (39) by letting $\nu \to 0$.

In equation (41) ω appears as a squared term only, while the right-hand side is real. Thus, the values of ω are either real or imaginary. When ω is imaginary, an arbitrary disturbance of the interface initially grows with time. Thus, the onset of instability is expressed through the dependence of ω^2 on the wavenumber k as well as on the applied electric field. Stability occurs when the electric term satisfies the following condition:

$$E < E^* = \frac{\varepsilon^{(1)} + \varepsilon^{(2)}}{k^2(\varepsilon^{(1)} - \varepsilon^{(2)})^2}[k^3\sigma_T - kg(\rho^{(1)} - \rho^{(2)}) + \nu^2(2k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)}))] \tag{42}$$

and the wavenumber satisfies the following condition:

$$k < \sqrt{\frac{\rho^{(1)} + \rho^{(2)}}{2(\eta^{(1)} + \eta^{(2)})}} \tag{43}$$

where $E^2 = E^{(1)}E^{(2)}$ is used. It is noted that the stability criterion (42) depends on the capillary gravity which affects the dielectric constants for the media and on viscoelasticity coefficients. The implication of viscosity coefficients will depend on the nonlinear stability criteria.

To develop the nonlinear effects for the amplitude modulation for the progressive waves, we need to go to the full equation (36). The treatment of the nonlinear equation (36) can be achieved through a perturbation scheme.

5. The perturbation scheme using the multiple scales method

To discuss the stabilization of the nonlinear system, we may introduce a modulation to the problem so that the linear dispersion relation D (ω, k) represents the slowly modulated wavetrain [33]. To do this, we may use the expansion procedure obtained formally by the method of multiple scaling [34]. The underlying idea of the method of multiple scales is to make the expansion representing the solution of the problem not a function of one independent variable only, but a function of two or more independent variables which are referred to as scales. We shall suppose a small parameter δ measures the ratio of a typical wavelength or periodic time relative to a typical length or time scale of the modulation (measures the steepness ratio as the perturbation parameter). The independent variables x and t, which are measured on the scale of the typical wavelength and time period, can be extended to introduce alternative independent variables

$$T_n = \delta^n T \qquad X_n = \delta^n X \qquad n = 0, 1, 2.$$

Thus, defining T_0, X_0 as the variables appropriate to fast variations and T_1, X_1, T_2, X_2 as the slow variables, the differential operators can now be expressed as the derivative expansions

$$\frac{\partial}{\partial T} = -\omega\frac{\partial}{\partial\theta} + \delta\frac{\partial}{\partial T_1} + \delta^2\frac{\partial}{\partial T_2} + \cdots$$
$$\frac{\partial}{\partial X} = k\frac{\partial}{\partial\theta} + \delta\frac{\partial}{\partial X_1} + \delta^2\frac{\partial}{\partial X_2} + \cdots$$

where $\theta = kX_0 - (\omega + \nu - k\mu^*)T_0$, T_0 and X_0 are the lowest order of the time and the phase of oscillations of the wavetrain. The analysis then follows the usual perturbation procedure and suppression of secular terms except that it is now more convenient to write the solution in complex form.

Now the operator L is extended to

$$L\left[-\mathrm{i}\omega, \mathrm{i}k + \delta\left(\frac{\partial}{\partial T_1}, \frac{\partial}{\partial X_1}\right) + \delta^2\left(\frac{\partial}{\partial T_2}, \frac{\partial}{\partial X_2}\right) + \cdots\right]. \tag{44}$$

The expression for the operator L can be expanded in powers of δ. This can be accomplished by using Taylor's theory about $(-\mathrm{i}\omega, \mathrm{i}k)$ and retains only terms up to $O(\delta^2)$. Thus,

$$L \to L_0 + \delta L_1 + \delta^2 L_2 + \cdots. \tag{45}$$

Expressing the expansion of the expanded operator (45) into equation (40) we get

$$(L_0 + \delta L_1 + \delta^2 L_2 + \cdots)\xi = 0.$$

Thus, it follows that

$$(D_0 + \delta D_1 + \delta^2 D_2 + \cdots)\gamma = 0 \tag{46}$$

where $D_0 = 0$ specifies the dispersion relation in the linear approximation

$$D_1 = \mathrm{i}\left\{\left[\frac{\partial D}{\partial\omega}\right]\frac{\partial}{\partial T_1} - \left[\frac{\partial D}{\partial k}\right]\frac{\partial}{\partial X_1}\right\} \tag{47}$$

$$D_2 = \mathrm{i}\left[\frac{\partial D}{\partial\omega}\right]\frac{\partial}{\partial T_2} - \mathrm{i}\left[\frac{\partial D}{\partial k}\right]\frac{\partial}{\partial X_2} - \frac{1}{2}\left[\frac{\partial^2 D}{\partial\omega^2}\right]\frac{\partial^2}{\partial T_1^2} - \frac{1}{2}\left[\frac{\partial^2 D}{\partial k^2}\right]\frac{\partial^2}{\partial X_1^2} + \left[\frac{\partial^2 D}{\partial\omega\partial k}\right]\frac{\partial^2}{\partial T_1\partial X_1}. \tag{48}$$

Now, when the nonlinearity is assumed to be of order (δ), ξ may be written in the form

$$\xi = \sum_{n=1}^{3}\delta^n\xi_n(\theta, X_1, X_2; T_1, T_2) + O(\delta^4). \tag{49}$$

Substituting (49) into (36), using (45) and equating like powers of δ on both sides we obtain the following three orders of δ:

$$L_0\xi_1 = 0 \tag{50}$$

$$L_0\xi_2 = -L_1\xi_1 + \alpha\xi_1^2 \tag{51}$$

$$L_0\xi_3 = -L_1\xi_2 - L_2\xi_1 + 2\alpha\xi_1\xi_2 + \beta\xi_1^3 \tag{52}$$

where

$$\alpha = k(\rho^{(1)} - \rho^{(2)} + 6k^2(\eta^{(1)} - \eta^{(2)}))(\omega + \nu)^2 - k^2\big[4k^2\big(\eta^{(1)}U_0^{(1)} - \eta^{(2)}U_0^{(2)}\big) + \big(\rho^{(1)}U_0^{(1)} - \rho^{(2)}U_0^{(2)}\big) - 2\mathrm{i}k(\mu^{(1)} - \mu^{(2)}) + 8k^2(\eta^{(1)} - \eta^{(2)})\mu^*\big](\omega + \nu) + k^3\left[2k^2\left(\eta^{(1)}U_0^{(1)^2} - \eta^{(2)}U_0^{(2)^2}\right) - 2\mathrm{i}k\big(\mu^{(1)}U_0^{(1)} - \mu^{(2)}U_0^{(2)}\big) + 4k^2(\eta^{(1)} - \eta^{(2)})\mu^{*^2} + \sigma_E\frac{\varepsilon^{(1)} - \varepsilon^{(2)}}{\varepsilon^{(1)} + \varepsilon^{(2)}}\right] \tag{53}$$

$$\beta = k^2\left\{(6k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)}))(\omega + \nu)^2 - k\big[4k^2\big(\eta^{(1)}U_0^{(1)} + \eta^{(2)}U_0^{(2)}\big) - \big(\rho^{(1)}U_0^{(1)} + \rho^{(2)}U_0^{(2)}\big) - 2\mathrm{i}k((\mu^{(1)} + \mu^{(2)}) + 4\mathrm{i}k(\eta^{(1)} + \eta^{(2)})\mu^*)\big](\omega + \nu) + \tfrac{1}{2}k\left[-k^2\sigma_T + 4g(\rho^{(1)} - \rho^{(2)}) - 4k^3\left(\left(\eta^{(1)}U_0^{(1)^2} + \eta^{(2)}U_0^{(2)^2}\right) + 2(\eta^{(1)} + \eta^{(2)})\mu^{*^2}\right)\right]\right\}. \tag{54}$$

These coefficients, in the case of the Rayleigh–Taylor model, are reduced to

$$\alpha_0 = k(\rho^{(1)} - \rho^{(2)} + 6k^2(\eta^{(1)} - \eta^{(2)}))\omega^2 + 2\mathrm{i}k^3(\mu^{(1)} - \mu^{(2)})\omega + k^3\sigma_E\frac{\varepsilon^{(1)} - \varepsilon^{(2)}}{\varepsilon^{(1)} + \varepsilon^{(2)}} \tag{55}$$

$$\beta_0 = k^2\left\{(6k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)}))\omega^2 + 2\mathrm{i}k^2(\mu^{(1)} + \mu^{(2)})\omega + \tfrac{1}{2}k[-k^2\sigma_T + 4g(\rho^{(1)} - \rho^{(2)})]\right\}. \tag{56}$$

We may now solve these equations in turn, noting that for each $n > 1$ up to two non-secular conditions may be obtained by setting both the θ-independent terms and the coefficient of $\exp(\mathrm{i}\theta)$ to zero in equations (50), (51) and (52). In the lowest order approximation we assume the following quasi-monochromatic wave as the solution of equation (50)

$$\xi_1 = \gamma(X_1, X_2; T_1, T_2)\exp(\mathrm{i}\theta) + \bar{\gamma}(X_1, X_2; T_1, T_2)\exp(-\mathrm{i}\theta)$$

where the amplitude γ is a complex function.

At $O(\delta^2)$, equation (51), in view of the solution (53), takes the form

$$L_0\xi_2 = -\mathrm{i}\left[\left(\frac{\partial D}{\partial\omega}\right)\frac{\partial\gamma}{\partial T_1} - \left(\frac{\partial D}{\partial k}\right)\frac{\partial\gamma}{\partial X_1}\right]\exp(\mathrm{i}\theta) + \alpha[\gamma^2\exp(2\mathrm{i}\theta) - \gamma\bar{\gamma}] + \text{c.c.} \tag{57}$$

Equation (56) contains secular terms, which correspond to the factor $\exp(\mathrm{i}\theta)$. The elimination of terms that lead to secular terms gives the following solvability condition:

$$\left(\frac{\partial D}{\partial\omega}\right)\frac{\partial\gamma}{\partial T_1} - \left(\frac{\partial D}{\partial k}\right)\frac{\partial\gamma}{\partial X_1} = 0 \tag{58}$$

and it is a complex conjugate relation. With this solvability condition, the solution of equation (51) is uniformly valid when

$$\xi_2 = \frac{\alpha}{\Omega}\gamma^2\exp(2\mathrm{i}\theta) + \text{c.c.} \tag{59}$$

The non-zero dominator Ω may be derived from the dispersion function $D(\omega, k)$ by replacing both ω and k with 2ω and $2k$, respectively.

At $O(\delta^3)$, we substitute (53) and (59) into the third-order equation (52); the requirement that the third-order solution is bounded yields the non-secularity condition

$$\mathrm{i}\left(\frac{\partial D}{\partial\omega}\right)\frac{\partial\gamma}{\partial T_2} - \mathrm{i}\left(\frac{\partial D}{\partial k}\right)\frac{\partial\gamma}{\partial X_2} - \frac{1}{2}\left(\frac{\partial^2 D}{\partial\omega^2}\right)\frac{\partial^2\gamma}{\partial T_1^2} - \frac{1}{2}\left(\frac{\partial^2 D}{\partial k^2}\right)\frac{\partial^2\gamma}{\partial X_1^2} + \left(\frac{\partial^2 D}{\partial\omega\partial k}\right)\frac{\partial^2\gamma}{\partial T_1\partial X_1} = \left(\frac{2\alpha^2}{\Omega} + 3\beta\right)\gamma^2\bar{\gamma}. \tag{60}$$

This is the solvability condition, which can be used to derive the nonlinear Schrödinger equation.

6. Derivation of the nonlinear Schrödinger equation

It is well known that the nonlinear Schrödinger equation is a generic equation describing unidirectional wave modulation. It has been shown to describe the spatial and temporal evolution of the envelope of a sinusoidal wave with phase $kx - \omega(k)t$, which draws potential energy from some background field. To derive the Schrödinger equation from the third-order solvability condition we may use the second-order solvability condition (58) and hence we obtain

$$\mathrm{i}\frac{\partial\gamma}{\partial\tau} + P\frac{\partial^2\gamma}{\partial\zeta^2} = (Q_r + \mathrm{i}Q_i)\gamma^2\bar{\gamma}. \tag{61}$$

In deriving this nonlinear equation, the following new slow variables are introduced for convenience:

$$\zeta = \delta^{-1}(X_2 - V_g T_2) = (X_1 - V_g T_1) = \delta(x - V_g t) \qquad \text{and} \qquad \tau = T_2 = \delta T_1 = \delta^2 t$$

where V_g is the group velocity given by $V_g = -\left(\frac{\partial D}{\partial k}\right)\left(\frac{\partial D}{\partial\omega}\right)^{-1}$.

The coefficients that appear in equation (61) are formulated as follows:

$$P = -\frac{1}{2}\left(V_g^2\frac{\partial^2 D}{\partial\omega^2} + 2V_g\frac{\partial^2 D}{\partial\omega\partial k} + \frac{\partial^2 D}{\partial k^2}\right)\left(\frac{\partial D}{\partial\omega}\right)^{-1} \tag{62}$$

$$Q_r + \mathrm{i}Q_i = \left(\frac{2\alpha^2}{\Omega} + 3\beta\right)\left[\frac{\partial D}{\partial\omega}\right]^{-1}. \tag{63}$$

Note that, in the case of the Rayleigh–Taylor instability, these coefficients are constructed as

$$\begin{aligned} P = \{&(g(\rho^{(1)} - \rho^{(2)}) + 2k\sigma_E - 3k^2\sigma_T)^2 + 8k^2(\eta^{(1)} + \eta^{(2)})(\sigma_E - 3k\sigma_T) \\ &- 2\omega^2(\rho^{(1)} + \rho^{(2)})[4(\eta^{(1)} + \eta^{(2)})\omega^2 \\ &+ 2(\sigma_E - 3k\sigma_T)]\}/8\omega^3[2k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)})] \end{aligned} \tag{64}$$

$$\begin{aligned} Q_r + \mathrm{i}Q_i = \Bigg\{ &k^2[6k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)})]\omega^2 + k^3\left[2g(\rho^{(1)} - \rho^{(2)}) - \frac{1}{2}k^2\sigma_T\right] \\ &+ 2\mathrm{i}k^4(\mu^{(1)} + \mu^{(2)})\omega + \frac{2k^2}{\Omega_0}\Bigg[[6k^2(\eta^{(1)} - \eta^{(2)}) + (\rho^{(1)} - \rho^{(2)})]\omega^2 \\ &+ 2\mathrm{i}k^2(\mu^{(1)} - \mu^{(2)})\omega + k^2\sigma_E\frac{\varepsilon^{(1)} - \varepsilon^{(2)}}{\varepsilon^{(1)} + \varepsilon^{(2)}}\Bigg]^2\Bigg\} \Bigg/ 2\omega[2k^2(\eta^{(1)} + \eta^{(2)}) \\ &- (\rho^{(1)} + \rho^{(2)})] \end{aligned} \tag{65}$$

where

$$\begin{aligned} \Omega_0 = 2k\{&(\rho^{(1)} + \rho^{(2)})[2k^2\sigma_T + g(\rho^{(1)} - \rho^{(2)})] + 2k^2(\eta^{(1)} + \eta^{(2)})[4k^2\sigma_T - 7g(\rho^{(1)} - \rho^{(2)}) \\ &- k\sigma_E]\}/[2k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)})]. \end{aligned} \tag{66}$$

The vanishing of the dominator Ω_0 refers to the second-harmonic resonance. In the absence of the viscoelasticity coefficient from (66), the result is reduced to that obtained earlier by Nayfeh [36]. In general, harmonic resonance may exist if (ω, k) and $(n\omega, nk)$ satisfy the same dispersion relation [34]. When resonance occurs, we find that both the surface distortion and the excited volume pulsation undergo modulation. At exact resonance, only amplitude modulations occur and the modulations are monotonic functions of time; the volume pulsation increases as it draws energy from the surface distortion mode. Near, but not at, resonance, energy is exchanged cyclically between the surface and volumetric modes; the oscillations in this case experience both amplitude and phase modulations. The phase modulation, which results in changes in the oscillation frequencies, has previously been observed and studied for oscillations of liquid drops. However, the second-harmonic resonance is obtained in the inviscid nonlinear Rayleigh–Taylor instability for both electrified fluids [18] and non-electrified fluids [36]. This second-harmonic resonance, in the inviscid case, occurs at the point

$$k = \sqrt{g(\rho^{(2)} - \rho^{(1)})/2\sigma_T}.$$

It appears that the influence of the electric fields at the nonlinear resonance case has no contribution to the value of this resonance point. This is in contrast to the case when the viscoelastic problem is considered. The resonance point in this case depends on the electric field as demonstrated from (66).

Equation (61) is derived formally from the nonlinear system (36), in which the coefficients P and $Q_r + \mathrm{i}Q_i$ are constructed in terms of the linear dispersion relation (39) which is of a complex nature. This equation is a standard nonlinear Schrödinger equation with complex coefficients and can be used to study the stability behaviour of the problem. If the solution of (61) is linearly perturbed, the perturbations are stable if both the following conditions are satisfied [34]:

$$Q_i = k^4 \left\{ \frac{4(\mu^{(1)} - \mu^{(2)})}{\Omega_0} \left[((\rho^{(1)} - \rho^{(2)}) + 6k^2(\eta^{(1)} - \eta^{(2)}))\omega^2 + k^2\sigma_E \left(\frac{\varepsilon^{(1)} - \varepsilon^{(2)}}{\varepsilon^{(1)} + \varepsilon^{(2)}} \right) \right] + (\mu^{(1)} + \mu^{(2)}) \right\} \Big/ [2k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)})] < 0 \tag{67}$$

$$\begin{aligned} PQ_r = {} & \{(g(\rho^{(1)} - \rho^{(2)}) + 2k\sigma_E - 3k^2\sigma_T)^2 + 8k^2(\eta^{(1)} + \eta^{(2)})(\sigma_E - 3k\sigma_T) \\ & - 2\omega^2(\rho^{(1)} + \rho^{(2)})[4(\eta^{(1)} + \eta^{(2)})\omega^2 + 2(\sigma_E - 3k\sigma_T)]\} \Big\{ k^2(6k^2(\eta^{(1)} + \eta^{(2)}) \\ & - (\rho^{(1)} + \rho^{(2)}))\omega^2 + k^3\left(2g(\rho^{(1)} - \rho^{(2)}) - \frac{1}{2}k^2\sigma_T\right) + \frac{2k^2}{\Omega_0}\Bigg[\Bigg(k^2\sigma_E\left(\frac{\varepsilon^{(1)} - \varepsilon^{(2)}}{\varepsilon^{(1)} + \varepsilon^{(2)}}\right) \\ & + ((\rho^{(1)} - \rho^{(2)}) + 6k^2(\eta^{(1)} - \eta^{(2)}))\omega^2\Bigg)^2 - 4k^4(\mu^{(1)} - \mu^{(2)})\omega^2\Bigg]\Big\} \Big/ \\ & 16\omega^4[2k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)})]^3 > 0. \end{aligned} \tag{68}$$

Otherwise, the system is unstable (i.e. the system does not oscillate about the steady state). Conditions (67) and (68), for stability, can be satisfied when $\Omega_0 > 0$, in which the wavenumber must satisfy condition (43). To this end, the requirements for stability, in terms of the electric field, are

For $\Omega_0 > 0$:

$$\begin{aligned} E^2 < {} & \frac{\varepsilon^{(1)} + \varepsilon^{(2)}}{k^2(\varepsilon^{(1)} - \varepsilon^{(2)})^2(\eta^{(1)} + \eta^{(2)})}\{(\rho^{(1)} + \rho^{(2)})[2k^2\sigma_T + g(\rho^{(1)} - \rho^{(2)})] \\ & + k(\eta^{(1)} + \eta^{(2)})[4k^4\sigma_T - 7g(\rho^{(1)} - \rho^{(2)})]\}. \end{aligned} \tag{69}$$

For $Q_i < 0$:

$$2kE^2\frac{(\varepsilon^{(1)}-\varepsilon^{(2)})^2}{\varepsilon^{(1)}+\varepsilon^{(2)}}\left\{\frac{\mu^{(1)}\varepsilon^{(2)}+\mu^{(2)}\varepsilon^{(2)}}{\varepsilon^{(1)}+\varepsilon^{(2)}}[2k(\eta^{(1)}+\eta^{(2)})-(\rho^{(1)}+\rho^{(2)})]\right.$$
$$\left.-8k^2(\mu^{(1)}\eta^{(1)}+\mu^{(2)}\eta^{(2)})-2k^2(\mu^{(1)}\eta^{(2)}+\mu^{(2)}\eta^{(1)})+(\mu^{(1)}\rho^{(2)}+\mu^{(2)}\rho^{(1)})\right\}$$
$$+2[k^2\sigma_T-g(\rho^{(1)}-\rho^{(2)})][8k^2(\mu^{(1)}\eta^{(1)}+\mu^{(2)}\eta^{(2)})+2k^2(\mu^{(1)}\eta^{(2)}+\mu^{(2)}\eta^{(1)})$$
$$-(\mu^{(1)}\rho^{(2)}+\mu^{(2)}\rho^{(1)})]-(\mu^{(1)}+\mu^{(2)})[4k^2\sigma_T-g(\rho^{(1)}-\rho^{(2)})]$$
$$\times[2k^2(\eta^{(1)}+\eta^{(2)})-(\rho^{(1)}+\rho^{(2)})]<0 \tag{70}$$

$$(a_2E^2+a_1E+a_0)(b_2E^2+b_1E+b_0)>0. \tag{71}$$

The constant coefficients a and b are as given in the appendix. The transition curves separating the stable region from the unstable region correspond to

$$Q_i = 0 \tag{72}$$

$$PQ_r = 0. \tag{73}$$

In addition to these transition curves, there is another transition curve, that is $\Omega_0 = 0$. Across this curve, both the conditions (67) and (68) change sign.

These marginal curves can be borne out by numerical estimation. Before proceeding with the numerical calculations for the stability profile, it is convenient to introduce a non-dimensional form. This dimensionless form is formulated such that the characteristic length $\ell = g/\omega^2$, the characteristic time $T = \omega^{-1}$ and the characteristic mass $M = \sigma_T/\omega^2$. Other dimensionless quantities are given by

$$k = k^*\left(\frac{\omega^2}{g}\right) \qquad \rho^{(j)} = \rho^{*(j)}\left(\frac{\sigma_T\omega^4}{g^3}\right) \qquad \mu^{(j)} = \mu^{*(j)}\left(\frac{\sigma_T\omega}{g}\right)$$

$$\eta^{(j)} = \eta^{*(j)}\left(\frac{\sigma_T}{g}\right) \qquad U_0^{(j)} = U_0^{*(j)}\left(\frac{g}{\omega}\right) \qquad E^{(j)} = E^{*(j)}\left(\frac{\sigma_T\omega^2}{\varepsilon^{(2)}g}\right)^{1/2}$$

where the asterisk refers to the dimensionless quantity. It will be omitted for simplicity.

7. Numerical estimation for stability configuration

In this section, the goal is to determine the numerical profiles for the stability pictures for surface waves propagating through an interface between two superposed viscoelastic fluids. In order to screen this examination, numerical calculations for linear stability condition (42) and the nonlinear stability criteria (67) and (68) are made for variation of the wavenumber k versus the product of the electric fields $E^2 = E^{(1)}E^{(2)}$. It follows that the plane (E–k) is partitioned by the transition curves that are used to separate the stable region from the unstable one.

In figure 2, the linear stability profile is illustrated, for a system which is assumed to be statically stable ($\rho^{(1)} < \rho^{(2)}$), in which the parameter E^2 is plotted against the wavenumber k. In this graph some variation for the upper viscoelastic coefficient $\eta^{(1)}$ (keeping the lower coefficient $\eta^{(2)}$ to unity) is shown for the purpose of comparison. To discuss the influence of the electric term on the stability picture, a specific case is considered of say $\eta^{(1)} = 0.5$. It appears that small values of the electric field have no implications for the stable system. The instability influence appears for some large values of E^2. It is observed that there is an unstable zone which is embedded in the whole stable region. This unstable zone is due to the

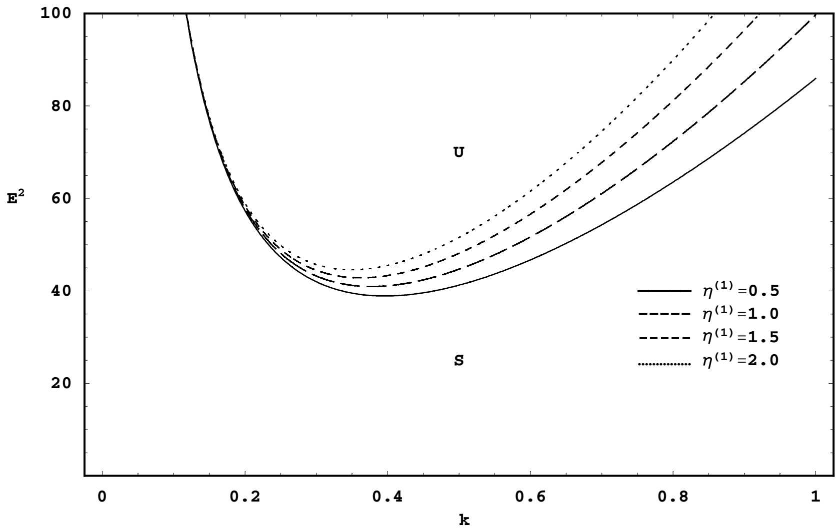

Figure 2. The stability diagram for the vertical electric term E^2 versus the wavenumber *k*. The calculation is made for a system of dimensionless form having $\rho^{(1)} = 0.75$, $\rho^{(2)} = 1$, $\eta^{(1)} = \eta^{(2)} = U_0^{(1)} = U_0^{(2)} = 1$ and $\varepsilon = 1.2$. The symbol *S* refers to the stable region while the symbol *U* indicates the unstable region.

application of the vertical electric field (see [7]). Thus, the vertical electric field is applied against the surface tension effect in the presence of the viscoelastic influence. Similarly, this destabilizing influence was demonstrated, earlier, by El-Dib and Matoog [26] and El-Dib [37] for viscoelastic fluids of Maxwell type and by El-Dib [38, 39] for viscoelastic fluids of Kelvin type. In contrast, the increase in the viscoelastic coefficient yields a construction in the unstable area, especially from the direction of increasing wavenumber *k*. This means that the increase in the viscoelastic coefficient has reduced the destabilizing influence for the application of the vertical field. On the other hand, one can say that, through the linear stability analysis, the viscoelastic coefficient has a stabilizing influence.

Further information about the influence of the vertical electric fields is demonstrated in figure 3. In the graph of figure 3, we discuss the influence of the dielectric ratio ε ($\varepsilon = \varepsilon^{(1)}/\varepsilon^{(2)}$) on the stability profile. With the help of the equilibrium condition (16), the ratio indicates the stratified electric field $E^{(2)}/E^{(1)}$ as well as the stratified dielectric constants $\varepsilon^{(1)} / \varepsilon^{(2)}$. Six consequent values for the ratio ε have been collected in this graph. It appears that as ε is increased, the transition curve moves up until it reaches the maximum state, continuing to increase ε moves the transition curve down. Thus, when the curve moves up it produces an increase in the stable region. However, the increase in the ratio ε plays a stabilizing as well as a destabilizing role. This means that there is a dual role for the ratio ε in the stability criterion. The contribution of the d.c. electric field to the stability conditions was discussed in detail by Mohamed and Elshehawey [18]. They found that if a finite amplitude disturbance is stable then a small modulation of the wave is stable. They reported that the electric field plays a dual role in the stability criterion. The dielectric constants of the fluids affected the stability conditions. For example, in a range of relatively smaller values of the density ratio, the field stabilizes or destabilizes depending on whether the lower fluid has a larger or smaller dielectric constant than the upper one. As the electric field increases, a greater stabilizing influence appears for a band of values of the density ratio.

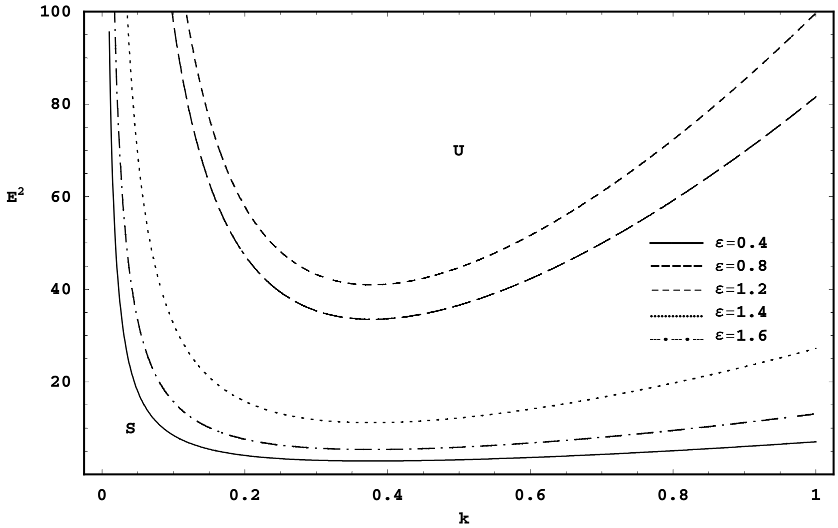

Figure 3. Variation of the ratio of the dielectric constant on the stability chart, for the same system as in figure 2.

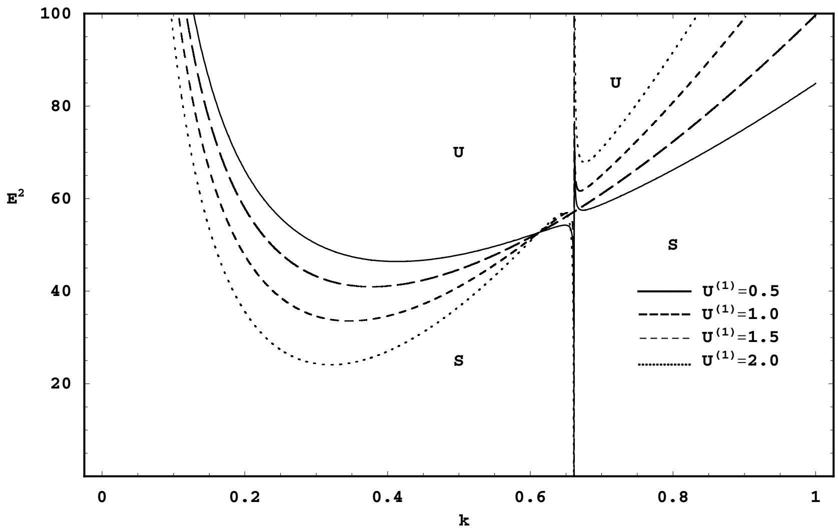

Figure 4. Diagram of the variation of the fluid velocity on the stability diagram, for the same system as in figure 2.

Figure 4 discusses the influence of fluid velocity on the linear stability profile. In this plane, some variation for the upper velocity $U_0^{(1)}$ is made for fixing the lower velocity to the value of $U_0^{(2)} = 1$. Inspection of the present plane reveals that the transition curve has moved down, in the region which is characterized by $k < \sqrt{(\rho^{(1)} + \rho^{(2)})/2(\eta^{(1)} + \eta^{(2)})}$. This means that there is a destabilizing influence, increasing fluid velocity. Another profile is observed for $k > \sqrt{(\rho^{(1)} + \rho^{(2)})/2(\eta^{(1)} + \eta^{(2)})}$, in which the transition curve moves up as $U_0^{(1)}$ is increased. At this stage, a stabilizing influence increasing the fluid velocity is presented.

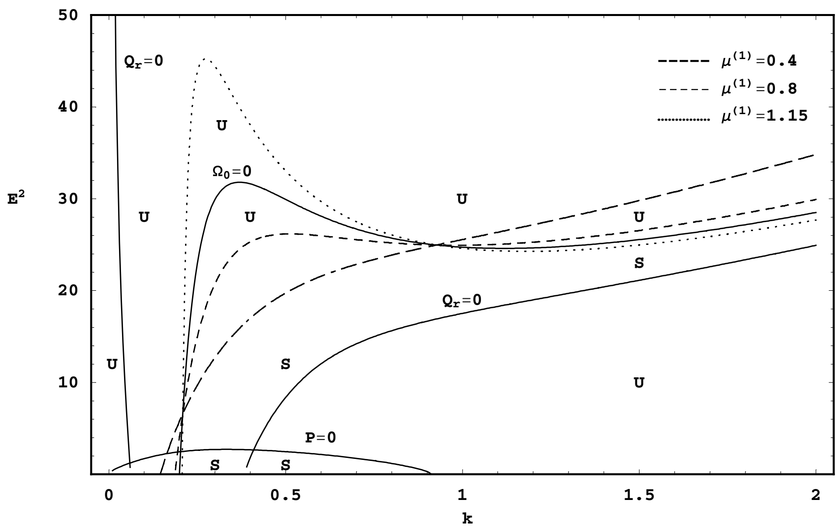

Figure 5. The stability diagram in view of the nonlinear behaviour, for a system having $\rho^{(1)} = 0.3$, $\rho^{(2)} = 1$, $\mu^{(1)} = \mu^{(2)} = \eta^{(1)} = \eta^{(2)}$ and $\varepsilon = 0.3$.

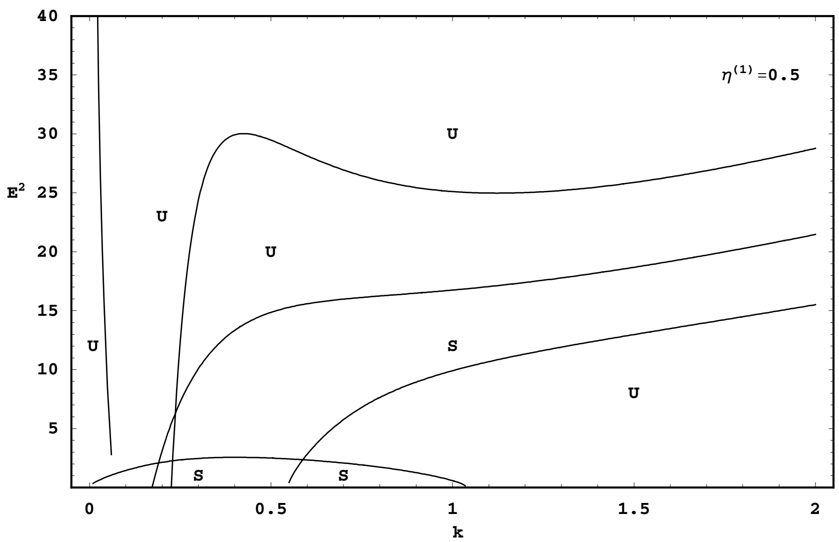

Figure 6. For the same system as in figure 5 except that $\eta^{(1)} = 0.5$.

It is convenient to note that there is no implication for the viscosity coefficients $\mu^{(j)}$ in the linear stability profile. Its implication is the subject of figure 5, in which computations for the nonlinear stability profile are made for stability criteria (67) and (68). The numerical calculations for the transition curves $\Omega_0 = 0$, $P = 0$, $Q_i = 0$ and $Q_r = 0$ are displayed in figures 5–8.

In figure 5, we collect three cases for the influence of the upper viscosity coefficient $\mu^{(1)}$. It is found from the numerical computations that the effective transition curve is the curve $Q_i = 0$. This curve has been affected by the increase in the viscosity coefficients.

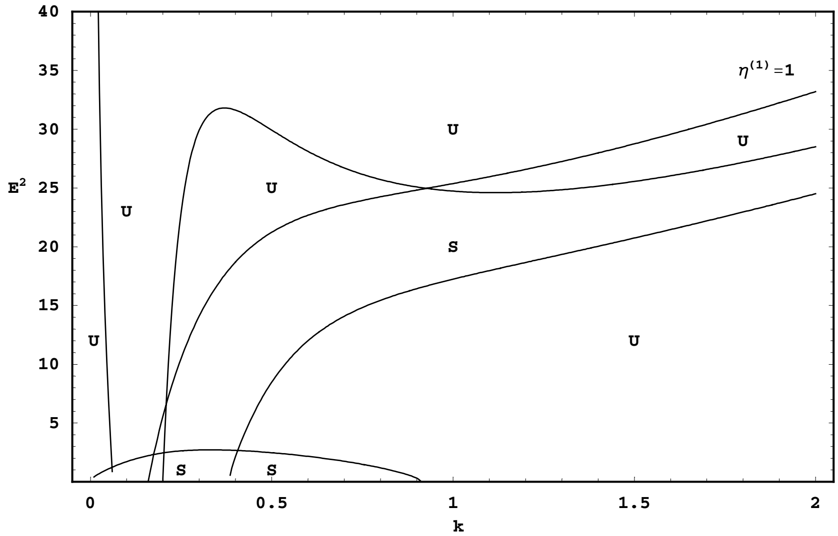

Figure 7. As in figure 5, except that $\eta^{(1)} = 1$.

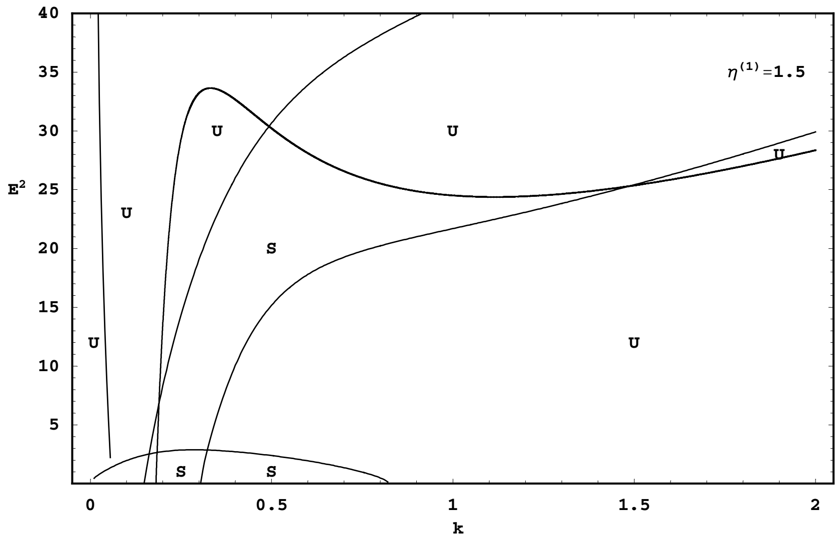

Figure 8. As in figure 5, except that $\eta^{(1)} = 1.5$.

Therefore, the remaining transition curves are plotted as solid curves. It appears that the increase in the viscosity coefficient causes the curve Q_i to shift up. Consequently, the stable area should increase. In the extraordinary case when $\mu^{(1)} = \mu^{(2)} = 1$ (i.e., the jump in the viscosity is zero) we found that the two curves $\Omega_0 = 0$ and $Q_i = 0$ coincide and hence the occurence of maximum stability. There is no stabilizing area which can be obtained for $\mu^{(1)} > 1$. This means that the damping role due to the viscosity is bounded.

The influence of the viscoelasticity coefficient on the nonlinear stability picture is the subject of figures 6–8. In these graphs the lower viscoelasticity is kept at $\eta^{(2)} = 1$, while three values for the upper viscoelastic coefficient are considered. Figure 6 deals with the case of $\eta^{(1)} = 0.5$, figure 7 is for the case of $\eta^{(1)} = 1$ and figure 8 is for the case of $\eta^{(1)} = 1.5$. Comparison of these figures reveals that the increase in the viscoelasticity coefficients yields a construction in the stable area. Thus, the destabilizing influence is due to the increase in $\eta^{(1)}$. This role is in contrast with the stabilizing influence discussed earlier in the linear stability profile (figure 2). This means that the viscoelasticity coefficient has changed its mechanism in the nonlinear stability criteria.

Appendix

The coefficients that appear in inequality (71) are

$$a_2 = \left(\frac{(\varepsilon^{(1)} - \varepsilon^{(2)})^2}{\varepsilon^{(1)} + \varepsilon^{(2)}}\right)^2 \Bigg\{2k^4(\eta^{(1)} + \eta^{(2)})[6k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)})] + k^2\Bigg[[6k^2(\eta^{(1)} - \eta^{(2)}) - (\rho^{(1)} - \rho^{(2)})] - \tfrac{1}{2}[2k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)})]\left(\frac{\varepsilon^{(1)} - \varepsilon^{(2)}}{\varepsilon^{(1)} + \varepsilon^{(2)}}\right)\Bigg]^2\Bigg\}$$

$$\begin{aligned} a_1 = \left(\frac{(\varepsilon^{(1)} - \varepsilon^{(2)})^2}{\varepsilon^{(1)} + \varepsilon^{(2)}}\right)\Bigg[&-2k^3(\eta^{(1)} + \eta^{(2)})\left\{2k^2(\eta^{(1)} + \eta^{(2)})\left[\tfrac{5}{2}k^2\sigma_T - g(\rho^{(1)} - \rho^{(2)})\right]\right.\\ &\left.- (\rho^{(1)} + \rho^{(2)})\left[\tfrac{1}{2}k^2\sigma_T + g(\rho^{(1)} - \rho^{(2)})\right]\right\} - k[6k^2(\eta^{(1)} + \eta^{(2)})\\ &- (\rho^{(1)} + \rho^{(2)})]\{(\rho^{(1)} + \rho^{(2)})[2k^2\sigma_T + g(\rho^{(1)} - \rho^{(2)})]\\ &+ 2k^2(\eta^{(1)} + \eta^{(2)})[4k^2\sigma_T - 7g(\rho^{(1)} - \rho^{(2)})]\} - 2k[k^2\sigma_T - g(\rho^{(1)} - \rho^{(2)})]\\ &\times [6k^2(\eta^{(1)} - \eta^{(2)}) + (\rho^{(1)} - \rho^{(2)})]\Bigg[[6k^2(\eta^{(1)} - \eta^{(2)}) - (\rho^{(1)} - \rho^{(2)})]\\ &- \tfrac{1}{2}[2k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)})]\left(\frac{\varepsilon^{(1)} - \varepsilon^{(2)}}{\varepsilon^{(1)} + \varepsilon^{(2)}}\right)\Bigg]\\ &+ 4k^4(\mu^{(1)} - \mu^{(2)})[2k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)})]\Bigg] \end{aligned}$$

$$\begin{aligned} a_0 = \{&(\rho^{(1)} + \rho^{(2)})[2k^2\sigma_T + g(\rho^{(1)} - \rho^{(2)})] + 2k^2(\eta^{(1)} + \eta^{(2)})[4k^2\sigma_T - 7g(\rho^{(1)} - \rho^{(2)})]\}\\ &\times \left\{2k^2(\eta^{(1)} + \eta^{(2)})\left[\tfrac{5}{2}k^2\sigma_T - g(\rho^{(1)} - \rho^{(2)})\right] - (\rho^{(1)} + \rho^{(2)})\left[\tfrac{1}{2}k^2\sigma_T\right.\right.\\ &\left.\left.+ g(\rho^{(1)} - \rho^{(2)})\right]\right\} + [6k^2(\eta^{(1)} - \eta^{(2)}) + \rho^{(1)} - \rho^{(2)}]^2[k^2\sigma_T - g(\rho^{(1)} - \rho^{(2)})]^2\\ &- 4k^3(\mu^{(1)} - \mu^{(2)})[k^2\sigma_T - g(\rho^{(1)} - \rho^{(2)})][2k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)})] \end{aligned}$$

$$b_2 = 4k^2(\rho^{(1)} + \rho^{(2)})[6k^2(\eta^{(1)} + \eta^{(2)}) + (\rho^{(1)} + \rho^{(2)})]$$

$$\begin{aligned} b_1 = -4k[&k^2\sigma_T - g(\rho^{(1)} - \rho^{(2)})][2k^4(\eta^{(1)} + \eta^{(2)})^2 + 4k^2(\eta^{(1)} + \eta^{(2)})(\rho^{(1)} + \rho^{(2)})\\ &+ (\rho^{(1)} + \rho^{(2)})^2] - 4k[2k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)})]\Big[4k^2 g(\rho^{(1)} - \rho^{(2)})\\ &\times (\eta^{(1)} + \eta^{(2)}) - 6k^4\sigma_T(\eta^{(1)} + \eta^{(2)}) - g\left(\rho^{(1)^2} - \rho^{(2)^2}\right)\Big] \end{aligned}$$

$$b_0 = 4k[2k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)})][k^2\sigma_T - g(\rho^{(1)} - \rho^{(2)})][2kg(\rho^{(1)} - \rho^{(2)})(\eta^{(1)} + \eta^{(2)}) - 3k\sigma_T(\rho^{(1)} + \rho^{(2)})] - [2k^2(\eta^{(1)} + \eta^{(2)}) - (\rho^{(1)} + \rho^{(2)})]^2[3k^2\sigma_T - g(\rho^{(1)} - \rho^{(2)})]^2 + 8k^2(\eta^{(1)} + \eta^{(2)})[4k^2(\eta^{(1)} + \eta^{(2)}) + (\rho^{(1)} + \rho^{(2)})][k^2\sigma_T - g(\rho^{(1)} - \rho^{(2)})]^2.$$

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